Sequences & Series

SL 1.2 | 1.3 | 1.4 | 1.8 | AHL 1.9

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Arithmetic AP · ADDS A FIXED D

$u_n = u_1 + (n-1)d$ booklet

n-th term — term 1 sits zero steps from the start.

$S_n = \frac{n}{2}\big(2u_1 + (n-1)d\big) = \frac{n}{2}(u_1 + u_n)$

Sum of n — second form when the last term is known.

$$d = u_{n+1} - u_n = \frac{u_n - u_m}{n - m}$$

Common difference — recover d from any two terms.

Geometric GP · TIMES A FIXED R

$u_n = u_1\, r^{\,n-1}$ booklet

n-th term — the exponent is $n-1$, not n.

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

Sum of n ($r \neq 1$); if $r = 1$ then $S_n = n\,u_1$.

$$r = \frac{u_{n+1}}{u_n} = \left(\frac{u_n}{u_m}\right)^{\frac{1}{n-m}}, \quad b^2 = ac \iff a, b, c \text{ GP}$$

Common ratio & the three-term geometric-mean test.

Infinite & convergence SL 1.8

$S_\infty = \frac{u_1}{1 - r}, \quad |r| < 1$ booklet

Sum to infinity — always state $|r| < 1$ first (a dedicated mark).

$$S_\infty - S_n = \frac{u_1\, r^n}{1 - r}$$

Tail (not in booklet) — "how many terms until within ε".

$$0.\overline{47} = \frac{47/100}{1 - 1/100} = \frac{47}{99}$$

Recurring decimals are infinite GPs.

Finance is a GP SL 1.4 · PAPER 2

$FV = PV\left(1 + \frac{r}{100k}\right)^{kn}$ booklet

Compound interest — ratio $\left(1 + \frac{r}{100k}\right)$, k periods/yr.

$$\text{Real} = PV\left(\frac{1 + i/100}{1 + j/100}\right)^n$$

Inflation-adjusted — the Finance Solver can't do this; use by hand.

$$N = k \times \text{years}$$

TVM solver — money out is negative; PV, FV opposite signs.

Sigma & standard results AHL · Σ NOTATION

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Standard sums — with $\sum r^3 = \left[\frac{n(n+1)}{2}\right]^2$.

$$\sum (a\,u_r + b\,v_r) = a\sum u_r + b\sum v_r, \quad \sum_{r=1}^{n} c = cn$$

Linearity — split and pull out constants.

$$\sum_{r=m}^{n} = \sum_{r=1}^{n} - \sum_{r=1}^{m-1}$$

Partial range — full minus the missing head.

Telescoping AHL · METHOD OF DIFFERENCES

$$\text{if } u_r = f(r) - f(r+1) \Rightarrow \sum_{r=1}^{n} u_r = f(1) - f(n+1)$$

Method of differences — the middle all cancels.

$$\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}, \quad \frac{1}{\sqrt{r} + \sqrt{r+1}} = \sqrt{r+1} - \sqrt{r}$$

Usual setups — partial fractions, or rationalise the surd.

"Show that … hence find $\sum \ldots$"

Exam pattern — prove the difference form, then telescope.

Linked AP & GP AHL

$a,\ a+md,\ a+kd$ in GP $\Rightarrow (a+md)^2 = a(a+kd)$

AP terms forming a GP — apply $b^2 = ac$, then solve for d.

$r = \dfrac{\text{2nd GP term}}{\text{1st}}, \qquad d = u_2 - u_1$

Recover the parameters once the shared terms are pinned.

Reject roots that break the problem ($d \neq 0$, $r \neq 1$, positivity).

Convergence range AHL · VARIABLE BASE

ratio $r = r(x) \Rightarrow$ converges $\iff |r(x)| < 1$

The condition IS an inequality in x — solve it for the range.

$1 + \ln x + (\ln x)^2 + \cdots, \quad r = \ln x$

Example — converges for $\frac{1}{e} < x < e$, then $S_\infty = \dfrac{1}{1-\ln x}$.

Trig bases too, e.g. $r = \cos\theta$ or $\cot x$ — solve $|r| < 1$ for the angle.

Key methods

- **Work back to n or k.** "Smallest n with $S_n > X$", or $u_k = 0$: form the equation, take logs, and **round up**.
- **Recover a sequence from** S_n. $u_1 = S_1$, $u_n = S_n - S_{n-1}$ for $n \geq 2$; $S_n = an^2 + bn$ is an AP with $d = 2a$.
- **Telescope.** Force u_r into $f(r) - f(r+1)$ (partial fractions / rationalising), then only the ends survive.
- **Linked AP–GP.** Write the GP condition on the AP terms, get one equation in d, solve, then read off r.
- **Variable-base convergence.** Identify $r(x)$, impose $|r(x)| < 1$, solve the inequality for the range of x.

Traps that cost marks

- It's $(n-1)d$, **not** nd — term 1 is the "zeroth jump".
- $u_n = S_n$ only when $n = 1$; otherwise $u_n = S_n - S_{n-1}$.
- State $|r| < 1$ **before** using S_∞ — the single most-dropped mark.
- Telescoping: line up $f(1) - f(n+1)$ carefully — an off-by-one loses the A mark.
- Convergence range: keep the modulus — $|r(x)| < 1$ often gives a two-sided interval.
- TVM solver: $N = k\times$ years (5-yr monthly loan is $N = 60$, not 5).

Worked example · derive $\sum r^2$ by telescoping HL · "show that ... hence"

Show that $\sum_{r=1}^{n}\left((r+1)^3 - r^3\right) = (n+1)^3 - 1$, *and hence show* $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

1. LHS telescopes: every $-r^3$ cancels the next $(r)^3$, leaving $(n+1)^3 - 1^3$. M1 A1
2. Expand the summand: $(r+1)^3 - r^3 = 3r^2 + 3r + 1$, so $3\sum r^2 + 3\cdot\frac{n(n+1)}{2} + n = (n+1)^3 - 1$. M1
3. Solve: $3\sum r^2 = (n+1)^3 - 1 - \frac{3n(n+1)}{2} - n \Rightarrow \sum r^2 = \frac{n(n+1)(2n+1)}{6}$. A1 AG

Worked example · linked AP & GP HL Paper 1 · 6 marks

The 1st, 4th and 16th terms of an arithmetic sequence ($u_1 = 15$, difference $d \neq 0$) are the first three terms of a geometric sequence. Find d and the ratio r.

1. Terms are $15,\ 15+3d,\ 15+15d$; the GP condition is $(15+3d)^2 = 15(15+15d)$. M1 A1
2. Expand: $225 + 90d + 9d^2 = 225 + 225d \Rightarrow 9d^2 = 135d \Rightarrow d = 15$ (as $d \neq 0$). M1 A1
3. $r = \dfrac{15 + 3(15)}{15} = \dfrac{60}{15} = 4$. A1